

# EXPONENTIAL ERGODICITY OF STOCHASTIC BURGERS EQUATIONS DRIVEN BY $\alpha$ -STABLE PROCESSES

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**ABSTRACT.** In this work, we prove the strong Feller property and the exponential ergodicity of stochastic Burgers equations driven by  $\alpha/2$ -subordinated cylindrical Brownian motions with  $\alpha \in (1, 2)$ . To prove the results, we truncate the nonlinearity and use the derivative formula for SDEs driven by  $\alpha$ -stable noises established in [33].

## 1. INTRODUCTION

Stochastic Burgers and Navier-Stokes equations, as models of studying the statistic theory of the turbulent fluid motion, has been studied in many literatures in past twenty years. In particular, the existence-uniqueness and ergodicity have been studied by many authors under non-degenerate or degenerate random perturbations (cf. [1, 6, 15, 13, 14, 17, 25] etc.). In these works, the random forces are assumed to be the Brownian noise, which can be naturally regarded as a continuous time model.

In recent years, the stochastic equations driven by Lévy type noises also attract much attention (cf. [9], [10], [22]-[24], [28]-[33], etc.). It was proved in [9] and [10] that there is a unique invariant measure for stochastic Burgers and 2D Navier-Stokes equations with Lévy noises. In these two works, the Lévy noises are assumed to be square integrable. This restriction clearly rules out the interesting  $\alpha$ -stable noises. It should be stressed that since the  $\alpha$ -stable noise exhibits the heavy tailed phenomenon, the stochastic equation driven by  $\alpha$ -stable processes recently causes great interest in physics (cf. [5, 20, 21, 30] etc.).

We shall consider in this paper the following stochastic Burgers equation on torus  $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ :

$$\partial_t u_t = u_t'' - u_t u_t' + \dot{\xi}_t, \quad (1.1)$$

where  $\dot{\xi}_t$  is some time-white noise. As mentioned above, when  $\dot{\xi}_t$  is additive Brownian noise, this type of equation has been intensively studied. In [1], Bertini, Cancrini and Jona-Lasini used Cole-Hopf's transformation to reduce equation (1.1) to a linear heat equation and obtained the existence of solutions. In [7], the ergodicity was also proved by using some truncation technique (see also [13, 14, 17] etc. for stochastic Navier-Stokes equations). In the present work we shall assume that  $\dot{\xi}_t$  is a type of  $\alpha$ -stable noise called  $\alpha/2$ -subordinated cylindrical Brownian noise and prove the exponential ergodicity of equation (1.1). There have been some results on ergodicity of stochastic systems driven by  $\alpha$ -stable type noises (cf. [32, 23, 19, 31]). In [19], Kulik obtained a nice criterion for the exponential mixing of a family of SDEs driven by  $\alpha$ -stable noises. We refer to [32] for the exponential mixing of stochastic spin systems with  $\alpha$ -stable noises, and to [23] for the exponential mixing of a family of semi-linear SPDEs with Lipschitz nonlinearity.

Let us now discuss the approach to the ergodicity. In a previous work [8], we have proved the existence of invariant measures for stochastic 2D Navier-Stokes equation by estimating the fractional moments. The proofs clearly also works for Burgers equation (1.1). To prove the exponential ergodicity, we shall use the Harris theorem (cf. [18]). Thus, the main task is to verify the conditions in Harris theorem, where an important step in our proof is to prove the

strong Feller property for truncated equation. It is well known that the truncating nonlinearity technique is a usual tool to establish the strong Feller property for Navier-Stokes and Ginzburg-Landau type equations ([13, 12, 25, 31]). To prove the strong Feller property, we shall truncate the quadratic nonlinearity of equation (1.1) and apply a derivative formula established in [33].

This paper is organized as follows: In Section 2, we give some necessary notions and notations. In particular, we study the stochastic convolutions in Hilbert space about the  $\alpha/2$ -subordinated cylindrical Brownian motions. In Section 3, we present a general result about the strong Feller property for SPDEs driven by  $\alpha/2$ -subordinated cylindrical Brownian motions. This result generalizes the corresponding one in [33, Theorem 4.1]. In Section 4, we prove our main result Theorem 4.2 by using suitable truncation technique and verifying the Harris conditions. In appendix, we study a deterministic Burgers equation and give some necessary dependence relation about the initial values. The result is by no means new. Since the proof is not so long, we include it here for the reader's convenience.

We conclude this section by introducing the following conventions: The letter  $C$  with or without subscripts will denote an unimportant constant, whose value may change in different occasions. Moreover, let  $\mathbb{U}$  be a Banach space, for  $R > 0$  we shall denote the ball in  $\mathbb{U}$  by

$$\mathbb{B}_R^{\mathbb{U}} := \{u \in \mathbb{U} : \|u\|_{\mathbb{U}} \leq R\}.$$

## 2. PRELIMINARIES

Let  $\mathbb{H}$  be a real separable Hilbert space with the inner product  $\langle \cdot, \cdot \rangle_0$ . The norm in  $\mathbb{H}$  is denoted by  $\|\cdot\|_0$ . Let  $A$  be a positive self-adjoint operator on  $\mathbb{H}$  with discrete spectral, i.e., there exists an orthogonal basis  $\{e_k\}_{k \in \mathbb{N}}$  and a sequence of real numbers  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty$  such that

$$Ae_k = \lambda_k e_k.$$

For  $\gamma \in \mathbb{R}$ , let  $\mathbb{H}^\gamma$  be the domain of the fractional operator  $A^{\frac{\gamma}{2}}$ , i.e.,

$$\mathbb{H}^\gamma := A^{-\frac{\gamma}{2}}(\mathbb{H}) = \left\{ \sum_k \lambda_k^{-\frac{\gamma}{2}} a_k e_k : (a_k)_{k \in \mathbb{N}} \subset \mathbb{R}, \sum_k a_k^2 < +\infty \right\},$$

with the inner product

$$\langle u, v \rangle_\gamma := \langle A^{\frac{\gamma}{2}} u, A^{\frac{\gamma}{2}} v \rangle_0 = \sum_k \lambda_k^\gamma \langle u, e_k \rangle_0 \langle v, e_k \rangle_0.$$

The semigroup associated to  $A$  is defined by

$$e^{-tA} u := \sum_k e^{-t\lambda_k} \langle u, e_k \rangle_0 e_k, \quad t \geq 0.$$

It is easy to see that for any  $\gamma > 0$ ,

$$\|A^\gamma e^{-tA} u\|_0 \leq \sup_{x>0} (x^\gamma e^{-x}) t^{-\gamma} \|u\|_0 = \gamma^\gamma e^{-\gamma} t^{-\gamma} \|u\|_0. \quad (2.1)$$

For a sequence of bounded real numbers  $\beta = (\beta_k)_{k \in \mathbb{N}}$ , let us define

$$Q_\beta : \mathbb{H} \rightarrow \mathbb{H}; \quad Q_\beta u := \sum_{k=1}^{\infty} \beta_k \langle u, e_k \rangle_0 e_k.$$

**Lemma 2.1.** *Suppose that for some  $\delta > 0$  and  $\theta, \theta' \in \mathbb{R}$  with  $\theta > \theta'$ ,*

$$\delta \lambda_k^{-\frac{\theta}{2}} \leq |\beta_k| \leq \delta^{-1} \lambda_k^{-\frac{\theta'}{2}}, \quad \forall k \in \mathbb{N}, \quad (2.2)$$

*Then we have*

$$\|A^{\frac{\theta'}{2}} Q_\beta u\|_0 \leq \delta^{-1} \|u\|_0, \quad u \in \mathbb{H}^0 \quad (2.3)$$

and

$$\|Q_\beta^{-1}u\|_0 \leq \delta^{-1} \|A^{\frac{\theta}{2}}u\|_0, \quad u \in \mathbb{H}^\theta. \quad (2.4)$$

*Proof.* By definition, we have

$$\|A^{\frac{\theta'}{2}}Q_\beta u\|_0^2 = \sum_k |\beta_k|^2 \lambda_k^{\theta'} \langle u, e_k \rangle_0^2 \leq \delta^{-2} \sum_k \langle u, e_k \rangle_0^2 = \delta^{-2} \|u\|_0^2,$$

and

$$\|Q_\beta^{-1}u\|_0^2 = \sum_{k=1}^{\infty} |\beta_k|^{-2} \langle u, e_k \rangle_0^2 \leq \delta^{-2} \sum_k \lambda_k^\theta \langle u, e_k \rangle_0^2 = \delta^{-2} \|A^{\frac{\theta}{2}}u\|_0^2.$$

The estimates follow.  $\square$

Let  $\{W_t^k, t \geq 0\}_{k \in \mathbb{N}}$  be a sequence of independent standard one-dimensional Brownian motion on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The cylindrical Brownian motion on  $\mathbb{H}$  is defined by

$$W_t := \sum_k W_t^k e_k.$$

For  $\alpha \in (0, 2)$ , let  $S_t$  be an independent  $\alpha/2$ -stable subordinator, i.e., an increasing one dimensional Lévy process with Laplace transform

$$\mathbb{E}e^{-\eta S_t} = e^{-t\eta^{\alpha/2}}, \quad \eta > 0.$$

The subordinated cylindrical Brownian motion  $\{L_t\}_{t \geq 0}$  on  $\mathbb{H}$  is defined by

$$L_t := W_{S_t}.$$

Notice that in general  $L_t$  does not belong to  $\mathbb{H}$ .

We recall the following estimate about the subordinator  $S_t$ .

**Lemma 2.2.** *We have*

$$\mathbb{P}(S_t \leq r) > 0, \quad r, t > 0, \quad (2.5)$$

and

$$\mathbb{E}(S_t^{-q}) \leq C t^{-\frac{2q}{\alpha}}, \quad q, t > 0. \quad (2.6)$$

*Proof.* Estimate (2.5) follows by the strict positivity of the distributional density  $p_t(s)$  of  $S_t$ . For (2.6), recalling that  $p_t(s)$  satisfies (cf. [3, (14)])

$$p_t(s) \leq C t s^{-1-\frac{\alpha}{2}} e^{-ts^{-\frac{\alpha}{2}}},$$

we have

$$\mathbb{E}(S_t^{-q}) \leq C \int_0^\infty t s^{-1-\frac{\alpha+2q}{2}} e^{-ts^{-\frac{\alpha}{2}}} ds = C t^{-\frac{2q}{\alpha}} \int_0^\infty u^{\frac{2q}{\alpha}} e^{-u} du,$$

where the last equality is due to the change of variable  $u = ts^{-\frac{\alpha}{2}}$ , and  $C$  only depends on  $\alpha, q$ .  $\square$

Let us now consider the following stochastic convolution:

$$Z_t := \int_0^t e^{-(t-s)A} Q_\beta dL_s = \sum_k \int_0^t e^{-(t-s)\lambda_k} \beta_k dW_{S_s}^k e_k,$$

where  $Q_\beta$  denotes the intensity of the noise. The following estimate about  $Z_t$  will play an important role in the next sections (cf. [24, 22]).

**Lemma 2.3.** Suppose that for some  $\gamma \in \mathbb{R}$ ,

$$K_\gamma := \sum_k \lambda_k^\gamma |\beta_k|^2 < +\infty. \quad (2.7)$$

Then for any  $p \in (0, \alpha)$  and  $T > 0$ ,

$$\sup_{t \in [0, T]} \mathbb{E} \|Z_t\|_{\gamma+1}^p \leq C_{\alpha, p} K_\gamma^{\frac{p}{2}} T^{\frac{p}{\alpha} - \frac{p}{2}}, \quad (2.8)$$

and for any  $\theta < \gamma$ ,

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|Z_t\|_\theta^p \right) \leq C_{\alpha, p} K_\gamma^{\frac{p}{2}} T^{\frac{p}{\alpha}} \left( 1 + T^{\frac{\gamma-\theta}{2}} \right), \quad (2.9)$$

and for any  $\varepsilon > 0$ ,

$$\mathbb{P} \left( \sup_{t \in [0, T]} \|Z_t\|_\theta \leq \varepsilon \right) > 0. \quad (2.10)$$

Moreover,  $t \mapsto Z_t$  is almost surely càdlàg in  $\mathbb{H}^\theta$ .

*Proof.* Estimate (2.8) follows by [33, Proposition 4.2]. Next, we prove (2.9). For any  $p \in (0, \alpha)$ , by Burkholder's inequality for Brownian motion, we have

$$\begin{aligned} \mathbb{E} \left( \sup_{t \in [0, T]} \|A^{\frac{\gamma}{2}} Q_\beta L_t\|_0^p \right) &= \mathbb{E} \left( \mathbb{E} \left( \sup_{t \in [0, T]} \|A^{\frac{\gamma}{2}} Q_\beta W_{\ell_t}\|_0^p \right) \middle| \ell=S \right) \\ &\leq \mathbb{E} \left( \mathbb{E} \left( \sup_{s \in [0, \ell_T]} \|A^{\frac{\gamma}{2}} Q_\beta W_s\|_0^p \right) \middle| \ell=S \right) \\ &\leq C_p \mathbb{E} \left( \|A^{\frac{\gamma}{2}} Q_\beta\|_{\text{H.S.}}^p S^{\frac{p}{2}} \right) \\ &= C_p \left( \sum_k \lambda_k^\gamma \beta_k^2 \right)^{\frac{p}{2}} \mathbb{E} \left( S^{\frac{p}{2}} \right) T^{\frac{p}{\alpha}}. \end{aligned} \quad (2.11)$$

In particular,

$$t \mapsto Q_\beta L_t \text{ is almost surely càdlàg in } \mathbb{H}^\gamma. \quad (2.12)$$

On the other hand, by integration by parts formula, we have

$$Z_t = Q_\beta L_t + \int_0^t A e^{-(t-s)A} Q_\beta L_s ds. \quad (2.13)$$

Hence, for any  $\theta < \gamma$ , by (2.1) we have

$$\begin{aligned} \|A^{\frac{\theta}{2}} Z_t\|_0 &\leq \|A^{\frac{\theta}{2}} Q_\beta L_t\|_0 + \int_0^t \|A^{1+\frac{\theta-\gamma}{2}} e^{-(t-s)A} A^{\frac{\gamma}{2}} Q_\beta L_s\|_0 ds \\ &\leq \lambda_1^{\theta-\gamma} \|A^{\frac{\gamma}{2}} Q_\beta L_t\|_0 + C \int_0^t \frac{\|A^{\frac{\gamma}{2}} Q_\beta L_s\|_0}{(t-s)^{1+\frac{\theta-\gamma}{2}}} ds \\ &\leq C \sup_{s \in [0, t]} \|A^{\frac{\gamma}{2}} Q_\beta L_s\|_0 \left( 1 + t^{\frac{\gamma-\theta}{2}} \right) =: \sup_{s \in [0, t]} \|A^{\frac{\gamma}{2}} Q_\beta L_s\|_0 \cdot \eta_t. \end{aligned} \quad (2.14)$$

Estimate (2.9) then follows by combining (2.11) and (2.14). Moreover, we also have that  $t \mapsto \int_0^t A e^{-(t-s)A} Q_\beta L_s ds$  is continuous in  $\mathbb{H}^\theta$ . Thus, the càdlàg property of  $t \mapsto Z_t$  in  $\mathbb{H}^\theta$  follows by (2.12) and (2.13).

Now, we prove (2.10). By (2.14) we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{t \in [0, T]} \|Z_t\|_\theta \leq \varepsilon\right) &\geq \mathbb{P}\left(\sup_{t \in [0, T]} \|A^{\frac{\gamma}{2}} Q_\beta W_{S_t}\|_0 \leq \varepsilon \eta_T^{-1}\right) \\
&\geq \mathbb{P}\left(\sup_{t \in [0, S_T]} \|A^{\frac{\gamma}{2}} Q_\beta W_t\|_0 \leq \varepsilon \eta_T^{-1}\right) \\
&\geq \mathbb{P}\left(\sup_{t \in [0, S_T]} \|A^{\frac{\gamma}{2}} Q_\beta W_t\|_0 \leq \varepsilon \eta_T^{-1}; S_T \leq 1\right) \\
&\geq \mathbb{P}\left(\sup_{t \in [0, 1]} \|A^{\frac{\gamma}{2}} Q_\beta W_t\|_0 \leq \varepsilon \eta_T^{-1}; S_T \leq 1\right) \\
&= \mathbb{P}\left(\sup_{t \in [0, 1]} \|A^{\frac{\gamma}{2}} Q_\beta W_t\|_0 \leq \varepsilon \eta_T^{-1}\right) \mathbb{P}(S_T \leq 1) > 0.
\end{aligned}$$

The last step is due to the fact that each term is positive.  $\square$

### 3. STRONG FELLER PROPERTY OF SPDEs DRIVEN BY SUBORDINATED CYLINDRICAL BROWNIAN MOTIONS

In this section, we consider the following general SPDE in Hilbert space  $\mathbb{H}$ :

$$du_t = [-Au_t + F(u_t)]dt + Q_\beta dL_t, \quad u_0 = \varphi \in \mathbb{H}, \quad (3.1)$$

where for some  $\delta > 0$  and  $\theta \geq \theta' \geq 0$ ,

$$\delta \lambda_k^{-\frac{\theta}{2}} \leq |\beta_k| \leq \delta^{-1} \lambda_k^{-\frac{\theta'}{2}}, \quad \forall k \in \mathbb{N}, \quad (3.2)$$

and for some  $\gamma, \gamma' \geq 0$ ,

$$F : \mathbb{H}^\gamma \rightarrow \mathbb{H}^{-\gamma'} \text{ is bounded and Lipschitz continuous.} \quad (3.3)$$

We need the following important constant:

$$\theta_0 := \inf \left\{ \theta > 0 : \sum_k \lambda_k^{-\theta} < +\infty \right\}. \quad (3.4)$$

The aim of this section is to prove that

**Theorem 3.1.** *Let  $\alpha \in (1, 2)$  and  $Z_t := \int_0^t e^{-(t-s)A} Q_\beta dL_s$ . Assume that (3.2) and (3.3) hold with*

$$\gamma - \theta' < 1 - \theta_0, \quad \gamma + \gamma' < 2, \quad (3.5)$$

*then for any  $\varphi \in \mathbb{H}$ , there exists a unique  $u_t = u_t(\varphi)$  satisfying that*

$$u_t - Z_t \in C([0, \infty); \mathbb{H}) \cap C((0, \infty); \mathbb{H}^\gamma),$$

*and*

$$u_t = e^{-tA} \varphi + \int_0^t e^{-(t-s)A} F(u_s) ds + Z_t. \quad (3.6)$$

*If in addition that for some  $\sigma \geq 0$ ,*

$$\gamma \leq \theta < \sigma + \frac{2}{\alpha}, \quad \theta + \gamma' < 2,$$

*then for any bounded Borel measurable function  $\Phi : \mathbb{H} \rightarrow \mathbb{R}$ ,  $\varphi_1, \varphi_2 \in \mathbb{H}^\sigma$  and  $t > 0$ ,*

$$|\mathbb{E}\Phi(u_t(\varphi_1)) - \mathbb{E}\Phi(u_t(\varphi_2))| \leq C_t t^{-\frac{1}{\alpha} - \frac{\theta + \sigma}{2}} \|\Phi\|_\infty \|\varphi_1 - \varphi_2\|_\sigma, \quad (3.7)$$

*where  $t \mapsto C_t$  is a continuous increasing function on  $[0, \infty)$ .*

*Proof.* The proof is divided into four steps.

(Step 1). We first establish the existence and uniqueness for (3.6). Set  $w_t := u_t - Z_t$ . Thus, to solve equation (3.6), it suffices to solve the following deterministic equation:

$$w_t = e^{-tA}\varphi + \int_0^t e^{-(t-s)A} F(w_s + Z_s) ds.$$

By (2.8), (3.2) and (3.5), we have

$$\int_0^T \mathbb{E} \|Z_t\|_\gamma^p dt \leq C_T K_{\gamma-1}^{\frac{p}{2}} < +\infty, \quad \forall T > 0,$$

where  $K_{\gamma-1}$  is defined by (2.7). Therefore, there exists a null set  $\Omega_0 \subset \Omega$  such that for all  $\omega \notin \Omega_0$ ,

$$Z_t(\omega) \in \mathbb{H}^\gamma \text{ for Lebesgue almost all } t \geq 0.$$

Below, we fix such an  $\omega$  and use the standard Picard's iteration argument to prove the existence. Define  $w_t^{(0)} := e^{-tA}\varphi$  and for  $n \in \mathbb{N}$ ,

$$w_t^{(n)} := e^{-tA}\varphi + \int_0^t e^{-(t-s)A} F(w_s^{(n-1)} + Z_s) ds. \quad (3.8)$$

By (2.1), we have

$$\begin{aligned} \|w_t^{(n)}\|_\gamma &\leq \|A^{\frac{\gamma}{2}} e^{-tA} \varphi\|_0 + \int_0^t \|A^{\frac{\gamma+\gamma'}{2}} e^{-(t-s)A} A^{-\frac{\gamma'}{2}} F(w_s^{(n-1)} + Z_s)\|_0 ds \\ &\leq C t^{-\frac{\gamma}{2}} \|\varphi\|_0 + C \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} \|F(w_s^{(n-1)} + Z_s)\|_{-\gamma'} ds \\ &\leq C t^{-\frac{\gamma}{2}} \|\varphi\|_0 + C \sup_{u \in \mathbb{H}^\gamma} \|F(u)\|_{-\gamma'} \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} ds \\ &= C t^{-\frac{\gamma}{2}} \|\varphi\|_0 + C t^{1-\frac{\gamma+\gamma'}{2}} \sup_{u \in \mathbb{H}^\gamma} \|F(u)\|_{-\gamma'}. \end{aligned} \quad (3.9)$$

Similarly, for any  $n, m \in \mathbb{N}$ , we also have

$$\begin{aligned} \|w_t^{(n)} - w_t^{(m)}\|_\gamma &\leq \int_0^t \|A^{\frac{\gamma+\gamma'}{2}} e^{-(t-s)A} A^{-\frac{\gamma'}{2}} (F(w_s^{(n-1)} + Z_s) - F(w_s^{(m-1)} + Z_s))\|_0 ds \\ &\leq C \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} \|F(w_s^{(n-1)} + Z_s) - F(w_s^{(m-1)} + Z_s)\|_{-\gamma'} ds \\ &\leq C \|F\|_{\text{Lip}} \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} \|w_s^{(n-1)} - w_s^{(m-1)}\|_\gamma ds, \end{aligned}$$

where  $\|F\|_{\text{Lip}} := \sup_{u \neq v \in \mathbb{H}^\gamma} \frac{\|F(u) - F(v)\|_{-\gamma'}}{\|u - v\|_\gamma}$ . This implies that for  $q < \frac{2}{\gamma+\gamma'}$ ,  $p = \frac{q}{q-1}$  and all  $t \in [0, T]$ ,

$$\begin{aligned} t^{\frac{\gamma}{2}} \|w_t^{(n)} - w_t^{(m)}\|_\gamma &\leq C t^{\frac{\gamma}{2}} \left( \int_0^t \left( (t-s)^{-\frac{\gamma+\gamma'}{2}} s^{-\frac{\gamma}{2}} \right)^q ds \right)^{\frac{1}{q}} \left( \int_0^t \left( s^{\frac{\gamma}{2}} \|w_s^{(n-1)} - w_s^{(m-1)}\|_\gamma \right)^p ds \right)^{\frac{1}{p}} \\ &\leq C t^{\frac{1}{q} - \frac{\gamma+\gamma'}{2}} \left( \int_0^t \left( s^{\frac{\gamma}{2}} \|w_s^{(n-1)} - w_s^{(m-1)}\|_\gamma \right)^p ds \right)^{\frac{1}{p}}. \end{aligned}$$

Thus, by (3.9) and Fatou's lemma, we have

$$\overline{\lim}_{n, m \rightarrow \infty} \sup_{s \in [0, t]} \left( s^{\frac{\gamma}{2}} \|w_s^{(n)} - w_s^{(m)}\|_\gamma \right)^p \leq C_T \int_0^t \overline{\lim}_{n, m \rightarrow \infty} \sup_{r \in [0, s]} \left( r^{\frac{\gamma}{2}} \|w_r^{(n-1)} - w_r^{(m-1)}\|_\gamma \right)^p ds.$$

By Gronwall's inequality, we obtain

$$\overline{\lim}_{n,m \rightarrow \infty} \sup_{s \in [0,T]} s^{\frac{\gamma}{2}} \|w_s^{(n)} - w_s^{(m)}\|_{\gamma} = 0. \quad (3.10)$$

Hence, there exists a  $w \in C((0, \infty); \mathbb{H}^{\gamma})$  such that for all  $T > 0$ ,

$$\overline{\lim}_{n \rightarrow \infty} \sup_{s \in [0,T]} s^{\frac{\gamma}{2}} \|w_s^{(n)} - w_s\|_{\gamma} = 0.$$

Taking limits for equation (3.8), we obtain the existence of a solution. The uniqueness follows from similar calculations.

(Step 2). Let  $\mathbb{H}_n$  be the finite dimensional subspace of  $\mathbb{H}$  spanned by  $\{e_1, \dots, e_n\}$ . Below we always use the isomorphism:

$$\mathbb{H}_n \simeq \mathbb{R}^n : u = \sum_{k=1}^n u_k e_k, \quad (u_1, \dots, u_n) \in \mathbb{R}^n.$$

Let  $\Pi_n$  be the projection operator from  $\mathbb{H}$  to  $\mathbb{H}_n$  defined by

$$\Pi_n u := \sum_{k=1}^n \langle u, e_k \rangle_{\mathbb{H}} e_k.$$

Let  $\rho_n$  be a sequence of nonnegative smooth functions with

$$\text{supp}(\rho_n) \subset \{z \in \mathbb{H}_n : |z| \leq 1/n\}, \quad \int_{\mathbb{H}_n} \rho_n(z) dz = 1.$$

Define

$$F_n(u) := \int_{\mathbb{H}_n} \rho_n(A^{\frac{\gamma}{2}}(u - z)) \Pi_n F(z) dz = \int_{\mathbb{H}_n} \rho_n(z) \Pi_n F(u - A^{-\frac{\gamma}{2}} z) dz, \quad u \in \mathbb{H}_n.$$

Then

$$A^{-\frac{\gamma'}{2}} F_n(u) = \int_{\mathbb{H}_n} \rho_n(u - z) \Pi_n A^{-\frac{\gamma'}{2}} F(z) dz.$$

Since  $F : \mathbb{H}^{\gamma} \rightarrow \mathbb{H}^{-\gamma'}$  is Lipschitz continuous, it is easy to see that

$$\sup_{u \in \mathbb{H}_n} \|\nabla_h A^{-\frac{\gamma'}{2}} F_n(u)\|_0 \leq \sup_{u \neq v} \frac{\|F(u) - F(v)\|_{-\gamma'}}{\|u - v\|_{\gamma}} \|A^{\frac{\gamma}{2}} h\|_0, \quad h \in \mathbb{H}_n. \quad (3.11)$$

Let

$$L_t^{(n)} := \sum_{k=1}^n W_{S_t}^k e_k.$$

Consider the following finite dimensional SDE:

$$du_t^{(n)} = [-Au_t^{(n)} + F_n(u_t^{(n)})]dt + Q_{\beta} dL_t^{(n)}, \quad u_0^{(n)} = \varphi \in \mathbb{H}_n.$$

By Duhamel's formula, we have

$$u_t^{(n)}(\varphi) = e^{-tA} \varphi + \int_0^t e^{-(t-s)A} F_n(u_s^{(n)}(\varphi)) ds + \int_0^t e^{-(t-s)A} Q_{\beta} dL_s^{(n)}.$$

It is easy to see that the directional derivative of  $\varphi \mapsto u_t^{(n)}(\varphi)$  along the direction  $h \in \mathbb{H}_n$  satisfies

$$\nabla_h u_t^{(n)}(\varphi) = e^{-tA} h + \int_0^t e^{-(t-s)A} \nabla_h (F_n \circ u_s^{(n)})(\varphi) ds$$

By (3.11), we further have

$$\|A^{\frac{\theta}{2}} \nabla_h u_t^{(n)}(\varphi)\|_0 \leq \|A^{\frac{\theta}{2}} e^{-tA} h\|_0 + \int_0^t \|A^{\frac{\theta+\gamma'}{2}} e^{-(t-s)A} \nabla_h A^{-\frac{\gamma'}{2}} (F_n \circ u_s^{(n)})(\varphi)\|_0 ds$$

$$\begin{aligned}
&\leq C t^{\frac{\sigma-\theta}{2}} \|A^{\frac{\sigma}{2}} h\|_0 + C \int_0^t (t-s)^{-\frac{\theta+\gamma'}{2}} \|\nabla_h A^{-\frac{\gamma'}{2}} (F_n \circ u_s^{(n)})(\varphi)\|_0 ds \\
&\leq C t^{\frac{\sigma-\theta}{2}} \|h\|_\sigma + C \int_0^t (t-s)^{-\frac{\theta+\gamma'}{2}} \|A^{\frac{\gamma}{2}} \nabla_h u_s^{(n)}(\varphi)\|_0 ds,
\end{aligned}$$

in view of  $\gamma \leq \theta$ , which implies that

$$t^{\frac{\theta-\sigma}{2}} \|A^{\frac{\theta}{2}} \nabla_h u_t^{(n)}(\varphi)\|_0 \leq C \|h\|_\sigma + C t^{\frac{\theta-\sigma}{2}} \int_0^t \left( (t-s)^{-\frac{\theta+\gamma'}{2}} s^{-\frac{\theta-\sigma}{2}} \right) \|A^{\frac{\theta}{2}} \nabla_h u_s^{(n)}(\varphi)\|_0 ds.$$

As in the proof of (3.10), we have

$$t^{\frac{\theta-\sigma}{2}} \|A^{\frac{\theta}{2}} \nabla_h u_t^{(n)}(\varphi)\|_0 \leq C_T \|h\|_\sigma, \quad h \in \mathbb{H}_n, \quad t \in (0, T], \quad (3.12)$$

where  $C_T$  is independent of  $n$ .

Now, by [33, Theorem 1.1], we have

$$\nabla_h \mathbb{E} \Phi(u_t^{(n)}(\varphi)) = \mathbb{E} \left( \Phi(u_t^{(n)}(\varphi)) \frac{1}{S_t} \int_0^t \langle Q_\beta^{-1} \nabla_h u_s^{(n)}(\varphi), dL_s^{(n)} \rangle_0 \right).$$

By Hölder's inequality, (2.6) and [33, Theorem 3.2], for any  $p \in (1, \alpha)$  and  $q = \frac{p}{p-1}$ , we have

$$\begin{aligned}
\|\nabla_h \mathbb{E} \Phi(u_t^{(n)}(\varphi))\|_0 &\leq \|\Phi\|_\infty \left( \mathbb{E} \left( \frac{1}{S_t^q} \right) \right)^{1/q} \left( \mathbb{E} \left| \int_0^t \langle Q_\beta^{-1} \nabla_h u_s^{(n)}(\varphi), dL_s^{(n)} \rangle_0 \right|^p \right)^{1/p} \\
&\leq C \|\Phi\|_\infty t^{-\frac{2}{\alpha}} \left( \int_0^t \mathbb{E} \|Q_\beta^{-1} \nabla_h u_s^{(n)}(\varphi)\|_0^\alpha ds \right)^{1/\alpha} \\
&\stackrel{(2.3)}{\leq} C \|\Phi\|_\infty t^{-\frac{2}{\alpha}} \left( \int_0^t \mathbb{E} \|A^{\frac{\theta}{2}} \nabla_h u_s^{(n)}(\varphi)\|_0^\alpha ds \right)^{1/\alpha} \\
&\stackrel{(2.4)}{\leq} C \|\Phi\|_\infty t^{-\frac{2}{\alpha}} \left( \int_0^t s^{\frac{(\sigma-\theta)\alpha}{2}} ds \right)^{1/\alpha} \|h\|_\sigma \\
&\leq C \|\Phi\|_\infty t^{-\frac{1}{\alpha} - \frac{\theta-\sigma}{2}} \|h\|_\sigma, \quad h \in \mathbb{H}_n.
\end{aligned}$$

From this, we in particular have

$$|\mathbb{E} \Phi(u_t^{(n)}(\varphi_1)) - \mathbb{E} \Phi(u_t^{(n)}(\varphi_2))| \leq C \|\Phi\|_\infty t^{-\frac{1}{\alpha} - \frac{\theta-\sigma}{2}} \|\varphi_1 - \varphi_2\|_\sigma, \quad \varphi_1, \varphi_2 \in \mathbb{H}_n, \quad (3.13)$$

where  $C$  is independent of  $n$ .

(Step 3). In this step we prove that for any fixed  $t > 0$  and  $\varphi \in \mathbb{H}^0$ ,

$$\lim_{n \rightarrow \infty} \|u_t^{(n)}(\Pi_n \varphi) - u_t(\varphi)\|_0 = 0, \quad P - a.s. \quad (3.14)$$

Set

$$Z_t^{(n)} := \int_0^t e^{-(t-s)A} Q_\beta dL_s^{(n)}, \quad w_t^{(n)} := u_t^{(n)} - Z_t^{(n)}.$$

Then

$$w_t^{(n)} - w_t = e^{-tA} (\Pi_n \varphi - \varphi) + \int_0^t e^{-(t-s)A} (F_n(w_s^{(n)} + Z_s^{(n)}) - F(w_s + Z_s)) ds,$$

and

$$\|w_t^{(n)} - w_t\|_\gamma \leq C t^{-\frac{\gamma}{2}} \|\Pi_n \varphi - \varphi\|_0 + C \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} \|F_n(w_s^{(n)} + Z_s^{(n)}) - F(w_s + Z_s)\|_{-\gamma'} ds.$$

Notice that by the definition of  $F_n$ ,

$$\|F_n(w_s^{(n)} + Z_s^{(n)}) - F(w_s + Z_s)\|_{-\gamma'} \leq \|F\|_{\text{Lip}} (\|w_s^{(n)} - w_s\|_\gamma + \|(\Pi_n - I)Z_s\|_\gamma + \frac{1}{n})$$



$$+ \|(\Pi_n - I)F(w_s + Z_s)\|_{-\gamma'}$$

and

$$\lim_{n \rightarrow \infty} \|(\Pi_n - I)Z_s\|_\gamma = 0, \quad \lim_{n \rightarrow \infty} \|(\Pi_n - I)F(w_s + Z_s)\|_{-\gamma'} = 0.$$

Since  $F$  is bounded, by Fatou's lemma, we obtain

$$\overline{\lim}_{n \rightarrow \infty} \|w_t^{(n)} - w_t\|_\gamma \leq C \int_0^t (t-s)^{-\frac{\gamma+\gamma'}{2}} \overline{\lim}_{n \rightarrow \infty} \|w_s^{(n)} - w_s\|_\gamma ds, \quad (3.15)$$

which then gives

$$\overline{\lim}_{n \rightarrow \infty} \|w_t^{(n)} - w_t\|_\gamma = 0 \quad (3.16)$$

as well as (3.14).

(Step 4). For proving (3.7), we first assume  $\Phi$  is continuous. In this case, by taking limits for (3.13), we obtain (3.7). For general bounded measurable  $\Phi$ , it follows by a standard approximation.  $\square$

#### 4. EXPONENTIAL ERGODICITY OF STOCHASTIC BURGERS EQUATIONS DRIVEN BY $\alpha$ -STABLE NOISES

We first recall the following abstract form of Harris' theorem (cf. [18, Theorem 4.2]).

**Theorem 4.1.** (Harris) *Let  $\mathcal{P}_t$  be a Markov semigroup over a Polish space  $\mathbb{X}$ . We assume that for some Lyapunov function  $V : \mathbb{X} \rightarrow \mathbb{R}_+$ ,*

(i) *there exist constants  $C_V, \gamma, K_V > 0$  such that for every  $x \in \mathbb{X}$  and  $t > 0$ ,*

$$\mathcal{P}_t V(x) \leq C_V e^{-\gamma t} V(x) + K_V;$$

(ii) *for every  $R > 0$ , there exists a time  $t > 0$  and  $\delta > 0$  such that for all  $x, y \in \mathbb{B}_R^\mathbb{X}$ ,*

$$\|\mathcal{P}_t(x, \cdot) - \mathcal{P}_t(y, \cdot)\|_{\text{TV}} := \sup_{\|\Phi\| \leq 1} |\mathcal{P}_t \Phi(x) - \mathcal{P}_t \Phi(y)| \leq 2 - \delta,$$

where  $\|\cdot\|_{\text{TV}}$  denotes the norm of total variation.

Then  $\mathcal{P}_t$  has a unique invariant probability measure  $\mu$  with

$$\|\mathcal{P}_t(x, \cdot) - \mu\|_{\text{TV}} \leq C e^{-\gamma_* t} (1 + V(x))$$

for some  $C, \gamma_* > 0$ .

In this section we shall use Theorems 3.1 and (4.1) to prove the exponential ergodicity of stochastic Burgers equations driven by  $\alpha$ -stable noises. Let  $\mathbb{H}$  be the space of all square integrable functions on the torus  $\mathbb{T} = [0, 2\pi)$  with vanishing mean values. Let  $Au = -u''$  be the second order differential operator. Then  $A$  is a positive self-adjoint operator on  $\mathbb{H}$ . Let  $\lambda_{2k} := \lambda_{2k+1} := k^2$  and

$$e_{2k}(x) := \pi^{-\frac{1}{2}} \cos(kx), \quad e_{2k+1}(x) := \pi^{-\frac{1}{2}} \sin(kx).$$

It is easy to see that  $\{e_k, k \in \mathbb{N}\}$  forms an orthogonal basis of  $\mathbb{H}$  and

$$Ae_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

In this case, let  $\theta_0$  be defined by (3.4), then

$$\theta_0 = \frac{1}{2}.$$

Define a bilinear operator

$$B(u, v) := uv', \quad u, v \in \mathbb{H}^1,$$

and write

$$B(u) = B(u, u).$$

Consider the following stochastic Burgers equation driven by  $L_t$ :

$$du_t = [-Au_t - B(u_t)]dt + Q_\beta dL_t, \quad u_0 = \varphi \in \mathbb{H}, \quad (4.1)$$

where  $Q_\beta$  denotes the intensity of the noise as above.

The main result of the paper is that

**Theorem 4.2.** *Let  $\alpha \in (1, 2)$ . Assume that for some  $\frac{3}{2} < \theta' \leq \theta < 2$  and  $\delta > 0$ ,*

$$\delta k^{-\theta} \leq |\beta_k| \leq \delta^{-1} k^{-\theta'}, \quad \forall k \in \mathbb{N}. \quad (4.2)$$

(i) *Let  $Z_t := \int_0^t e^{-(t-s)A} Q_\beta dL_s$ . Then for any  $\varphi \in \mathbb{H}$ , there exists a unique  $u_t(\varphi)$  with*

$$u_t - Z_t \in C([0, \infty), \mathbb{H}) \cap C((0, \infty), \mathbb{H}^1)$$

*solving equation (4.1). In particular,  $(t, \varphi) \mapsto u_t(\varphi)$  is a Markov process on  $\mathbb{H}$ . We write*

$$\mathcal{P}_t \Phi(\varphi) := \mathbb{E} \Phi(u_t(\varphi)).$$

(ii)  *$(\mathcal{P}_t)_{t>0}$  is strong Feller, i.e., for any bounded measurable function  $\Phi$  on  $\mathbb{H}$  and  $t > 0$ ,  $\mathcal{P}_t \Phi$  is a continuous function on  $\mathbb{H}$ .*

(iii) *There exists a unique invariant probability measure  $\mu$  on  $\mathbb{H}$  such that*

$$\|\mathcal{P}_t(\varphi, \cdot) - \mu\|_{TV} \leq C e^{-\gamma_* t} (1 + \|\varphi\|_0) \quad (4.3)$$

*for some  $C, \gamma_* > 0$ .*

*Proof.* We divide the proof into four steps.

(Step 1). In view of  $\theta' > \frac{3}{2}$ , for any  $\gamma \in (1, \theta' - \frac{1}{2})$ , by (2.9) we have

$$\mathbb{E} \left( \sup_{t \in [0, T]} \|Z_t\|_1 \right) \leq C_\alpha \left( \sum_k k^{2\gamma - 2\theta'} \right) T^{\frac{1}{\alpha}} \left( 1 + T^{\frac{\gamma-1}{2}} \right) < +\infty, \quad T > 0. \quad (4.4)$$

Thus, (i) follows by Theorem 5.1 below.

(Step 2). In this step, we prove the following claim: For given  $R > 0$ , there exist  $T = T(R) \in (0, 1]$  and  $K_1 = K_1(R, T) > 0$ ,  $K_2 = K_2(\alpha, \theta') > 0$  such that for any bounded measurable function  $\Phi$  on  $\mathbb{H}$ ,  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}^1}$  and all  $t \in (0, T]$ ,

$$|\mathcal{P}_t \Phi(\varphi_1) - \mathcal{P}_t \Phi(\varphi_2)| \leq K_1 t^{-\frac{1}{\alpha} - \frac{\theta-1}{2}} \|\Phi\|_\infty \|\varphi_1 - \varphi_2\|_1 + \frac{K_2 t^{\frac{1}{\alpha}}}{R}. \quad (4.5)$$

Consider the following truncated equation:

$$du_t^R = [-Au_t^R - B_R(u_t^R)]dt + Q_\beta dL_t, \quad u_0^R = \varphi \in \mathbb{H}^1,$$

where

$$B_R(u) := B(u) \cdot \chi(\|u\|_1 / (5R)),$$

and  $\chi \in C^\infty(\mathbb{R}, [0, 1])$  satisfies

$$\chi(r) = 1, \quad \forall |r| \leq 1; \quad \chi(r) = 0, \quad \forall |r| > 2.$$

Define the stopping time

$$\tau_\varphi^R(\omega) := \inf \{ t > 0 : \|u_t(\varphi; \omega)\|_1 \geq 5R \}, \quad \varphi \in \mathbb{B}_R^{\mathbb{H}^1},$$

and let

$$w_t(\varphi; \omega) = u_t(\varphi; \omega) - Z_t(\omega).$$

Then we have

$$\mathbb{P}(\tau_\varphi^R \leq t) = \mathbb{P} \left( \sup_{s \in [0, t]} \|u_s(\varphi)\|_1 \geq 5R \right) \leq \mathbb{P} \left( \sup_{s \in [0, t]} \|w_s(\varphi)\|_1 + \sup_{s \in [0, t]} \|Z_s\|_1 \geq 5R \right)$$

$$\leq \mathbb{P} \left( \sup_{s \in [0, t]} \|w_s(\varphi)\|_1 \geq 4R, \sup_{s \in [0, t]} \|Z_s\|_1 \leq R \right) + \mathbb{P} \left( \sup_{s \in [0, t]} \|Z_s\|_1 > R \right). \quad (4.6)$$

By Theorem 5.1 below, there exists a time  $T = T(R) \in (0, 1]$  such that for any  $\varphi \in \mathbb{B}_R^{\mathbb{H}^1}$  and  $t \in (0, T]$ , if  $\sup_{s \in [0, t]} \|Z_s(\omega)\|_1 \leq R$ , then

$$\sup_{s \in [0, t]} \|w_s(\varphi; \omega)\|_1 \leq 3R. \quad (4.7)$$

Hence, by (4.6) and Chebychev's inequality, we have for any  $\varphi \in \mathbb{B}_R^{\mathbb{H}^1}$ ,

$$\mathbb{P}(\tau_\varphi^R \leq t) \leq \mathbb{P} \left( \sup_{s \in [0, t]} \|Z_s\|_1 > R \right) \leq \frac{\mathbb{E} \left( \sup_{s \in [0, t]} \|Z_s\|_1 \right)}{R} \stackrel{(4.4)}{\leq} \frac{C_{\alpha, \theta'} t^{\frac{1}{\alpha}}}{R}. \quad (4.8)$$

On the other hand, by the uniqueness of solutions, we have

$$u_t(\varphi) = u_t^R(\varphi), \forall t \in [0, \tau_\varphi^R].$$

Thus, if we choose  $\sigma = \gamma = 1$  and  $\gamma' = 0$  in Theorem 3.1, then by (3.7), we have for any  $t \in (0, T]$  and  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}^1}$ ,

$$\begin{aligned} |\mathcal{P}_t \Phi(\varphi_1) - \mathcal{P}_t \Phi(\varphi_2)| &\leq \left| \mathbb{E}(\Phi(u_t(\varphi_1)); \tau_{\varphi_1}^R > t) - \mathbb{E}(\Phi(u_t(\varphi_2)); \tau_{\varphi_2}^R > t) \right| + \mathbb{P}(\tau_{\varphi_1}^R \leq t) + \mathbb{P}(\tau_{\varphi_2}^R \leq t) \\ &= \left| \mathbb{E}(\Phi(u_t^R(\varphi_1)); \tau_{\varphi_1}^R > t) - \mathbb{E}(\Phi(u_t^R(\varphi_2)); \tau_{\varphi_2}^R > t) \right| + \mathbb{P}(\tau_{\varphi_1}^R \leq t) + \mathbb{P}(\tau_{\varphi_2}^R \leq t) \\ &\leq \left| \mathbb{E}(\Phi(u_t^R(\varphi_1))) - \mathbb{E}(\Phi(u_t^R(\varphi_2))) \right| + 2\mathbb{P}(\tau_{\varphi_1}^R \leq t) + 2\mathbb{P}(\tau_{\varphi_2}^R \leq t) \\ &\leq K_1 t^{-\frac{1}{\alpha} - \frac{\theta-1}{2}} \|\Phi\|_\infty \|\varphi_1 - \varphi_2\|_1 + 2\mathbb{P}(\tau_{\varphi_1}^R \leq t) + 2\mathbb{P}(\tau_{\varphi_2}^R \leq t), \end{aligned}$$

which together with (4.8) gives (4.5).

(Step 3). In this step, we prove (ii). Let  $\Phi$  be a bounded measurable function on  $\mathbb{H}$ . Let us first show that for any  $t > 0$ ,

$$\varphi \mapsto \mathcal{P}_t \Phi(\varphi) \text{ is continuous on } \mathbb{H}^1. \quad (4.9)$$

Let  $\{\varphi_n\} \subset \mathbb{B}_R^{\mathbb{H}^1}$  converge to  $\varphi$  in  $\mathbb{H}^1$ . Let  $T = T(R) \in (0, 1]$  be as in Step 2. For fixed  $t > 0$ , by (4.5) we have

$$\begin{aligned} |\mathcal{P}_t \Phi(\varphi_n) - \mathcal{P}_t \Phi(\varphi)| &= |\mathcal{P}_{t \wedge T} \mathcal{P}_{t-t \wedge T} \Phi(\varphi_n) - \mathcal{P}_{t \wedge T} \mathcal{P}_{t-t \wedge T} \Phi(\varphi)| \\ &\leq K_1 (t \wedge T)^{-\frac{1}{\alpha} - \frac{\theta-1}{2}} \|\mathcal{P}_{t-t \wedge T} \Phi\|_\infty \|\varphi_n - \varphi\|_1 + \frac{K_2 (t \wedge T)^{\frac{1}{\alpha}}}{R} \\ &\leq K_1 (t \wedge T)^{-\frac{1}{\alpha} - \frac{\theta-1}{2}} \|\Phi\|_\infty \|\varphi_n - \varphi\|_1 + \frac{K_2}{R}, \end{aligned}$$

where  $K_1 = K_1(R, T)$  and  $K_2 = K_2(\alpha, \theta')$ . First letting  $n \rightarrow \infty$  and then  $R \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} |\mathcal{P}_t \Phi(\varphi_n) - \mathcal{P}_t \Phi(\varphi)| = 0.$$

Next we prove that

$$\varphi \mapsto \mathcal{P}_t \Phi(\varphi) \text{ is continuous on } \mathbb{H}. \quad (4.10)$$

For  $R > 0$ , define

$$\Omega_R := \left\{ \omega : \sup_{s \in [0, 1]} \|Z_s(\omega)\|_1 \leq R \right\}.$$

By Theorem 5.1 again, there exists a time  $T = T(R) \in (0, 1)$  such that for any  $\omega \in \Omega_R$  and all  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}}$  and  $t \in (0, T]$ ,

$$\|u_t(\varphi_1; \omega) - u_t(\varphi_2; \omega)\|_1 = \|w_t(\varphi_1; \omega) - w_t(\varphi_2; \omega)\|_1 \leq 2t^{-\frac{1}{2}}\|\varphi_1 - \varphi_2\|_0. \quad (4.11)$$

Let  $\{\varphi_n\} \subset \mathbb{B}_R^{\mathbb{H}}$  converge to  $\varphi$  in  $\mathbb{H}$ . For any  $t > 0$ , we have

$$\begin{aligned} |\mathcal{P}_t\Phi(\varphi_n) - \mathcal{P}_t\Phi(\varphi)| &= \left| \mathbb{E}\left((\mathcal{P}_{t-t\wedge T}\Phi)(u_{t\wedge T}(\varphi_n))\right) - \mathbb{E}\left((\mathcal{P}_{t-t\wedge T}\Phi)(u_{t\wedge T}(\varphi))\right) \right| \\ &\leq \left| \mathbb{E}\left((P_{t-t\wedge T}\Phi)(u_{t\wedge T}(\varphi_n)) - (P_{t-t\wedge T}\Phi)(u_{t\wedge T}(\varphi)); \Omega_R\right) \right| + 2\mathbb{P}(\Omega_R^c), \end{aligned}$$

which together with (4.9) and (4.11) yields (4.10) by first letting  $n \rightarrow \infty$  and then  $R \rightarrow \infty$ .

(Step 4). In this step, we prove (iii). Take  $V(\varphi) = 1 + \|\varphi\|_0$ . Let us first check (i) of Theorem 4.1. Arguing as deriving (1.2) of [8] and taking  $\theta = 1$  therein, we have

$$\mathbb{E}\left(\sup_{s \in [0, t]} \|u_s\|_0\right) + \mathbb{E}\left(\int_0^t \frac{\|u_s\|_1^2}{(\|u_s\|_0^2 + 1)^{1/2}} ds\right) \leq C(1 + \|\varphi\|_0 + t).$$

which, together with the spectral gap inequality  $\|u\|_0 \leq \|u\|_1$ , implies

$$\mathbb{E}(\|u_t\|_0 + 1) + \mathbb{E}\left(\int_0^t \frac{\|u_s\|_0^2 + 1}{(\|u_s\|_0^2 + 1)^{1/2}} ds\right) \leq C(1 + \|\varphi\|_0 + t).$$

From this, we get

$$\mathbb{E}V(u_t) \leq -\frac{1}{2} \int_0^t \mathbb{E}V(u_s) ds + CV(\varphi) + Ct,$$

which implies that

$$\mathbb{E}V(u_t) \leq Ce^{-\frac{1}{2}t}V(\varphi) + 2C, \quad \forall t > 0. \quad (4.12)$$

Next we check (ii) of Theorem 4.1. Fix  $R > 0$ . Let  $\varepsilon, t_0 > 0$ , to be determined later. Define

$$\Omega_{t_0}^\varepsilon := \left\{ \omega : \sup_{s \in [0, t_0+1]} \|Z_s(\omega)\|_1 \leq \varepsilon \right\}.$$

By (5.6) below, one can choose  $\varepsilon_0 := \frac{1}{2C_1} \wedge \frac{1}{\sqrt{2C_2+1}}$  and  $t_0 \geq 2 \log(R^2/\varepsilon^4)$  so that for each  $\varepsilon \in (0, \varepsilon_0]$ , all  $\omega \in \Omega_{t_0}^\varepsilon$ ,  $\varphi \in \mathbb{B}_R^{\mathbb{H}}$  and  $t \in [t_0, t_0 + 1]$ ,

$$\begin{aligned} \|w_t(\varphi, \omega)\|_0^2 &\leq \|\varphi\|_0^2 e^{(C_1\varepsilon-1)t} + C_2\varepsilon^4 \int_0^t e^{(C_1\varepsilon-1)(t-s)} ds \\ &\leq R^2 e^{-t/2} + 2C_2\varepsilon^4 \leq (2C_2 + 1)\varepsilon^4 \leq \varepsilon^2. \end{aligned}$$

By using Theorem 5.1 again with  $R = \varepsilon$  and starting from  $t_0$  therein, there exists a time  $t_1 \in (t_0, t_0 + 1]$  such that for all  $t \in (t_0, t_1]$  and  $\varepsilon \in (0, \varepsilon_0]$ ,

$$\|w_t(\varphi, \omega)\|_1 \leq (t - t_0)^{-\frac{1}{2}}(2\|w_{t_0}(\varphi, \omega)\|_0) \leq (t - t_0)^{-\frac{1}{2}}(2\varepsilon).$$

In particular, for each  $\varepsilon \in (0, \varepsilon_0)$ , all  $\omega \in \Omega_{t_0}^\varepsilon$  and  $\varphi \in \mathbb{B}_R^{\mathbb{H}}$ ,

$$\|w_{t_1}(\varphi, \omega)\|_1 \leq 2(t_1 - t_0)^{-\frac{1}{2}}\varepsilon.$$

For  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}}$ , if we define

$$A_{t_1}^\varepsilon(\varphi_1, \varphi_2) := \left\{ \omega : \|w_{t_1}(\varphi_1, \omega)\|_1 + \|w_{t_1}(\varphi_2, \omega)\|_1 \leq 4(t_1 - t_0)^{-\frac{1}{2}}\varepsilon \right\},$$

then from the above implication, one has

$$\Omega_{t_0}^\varepsilon \subset A_{t_1}^\varepsilon(\varphi_1, \varphi_2). \quad (4.13)$$

Now by definition, for any  $t_2 \in (t_1, t_0 + 1)$  with  $t_2 - t_1$  being small, we have

$$\begin{aligned}
\|\mathcal{P}_{t_2}(\varphi_1, \cdot) - \mathcal{P}_{t_2}(\varphi_2, \cdot)\|_{\text{TV}} &:= \sup_{\|\Phi\|_\infty \leq 1} |\mathcal{P}_{t_2}\Phi(\varphi_1) - \mathcal{P}_{t_2}\Phi(\varphi_2)| \\
&= \sup_{\|\Phi\|_\infty \leq 1} |\mathcal{P}_{t_1}\mathcal{P}_{t_2-t_1}\Phi(\varphi_1) - \mathcal{P}_{t_1}\mathcal{P}_{t_2-t_1}\Phi(\varphi_2)| \\
&= \sup_{\|\Phi\|_\infty \leq 1} \left| \mathbb{E}(\mathcal{P}_{t_2-t_1}\Phi(u_{t_1}(\varphi_1)) - \mathcal{P}_{t_2-t_1}\Phi(u_{t_1}(\varphi_2))) \right| \\
&\leq \sup_{\|\Phi\|_\infty \leq 1} \left| \mathbb{E}(\mathcal{P}_{t_2-t_1}\Phi(u_{t_1}(\varphi_1)) - \mathcal{P}_{t_2-t_1}\Phi(u_{t_1}(\varphi_2)); A_{t_1}^\varepsilon(\varphi_1, \varphi_2)) \right| \\
&\quad + 2(1 - \mathbb{P}(A_{t_1}^\varepsilon(\varphi_1, \varphi_2))).
\end{aligned}$$

Noticing that on  $A_{t_1}^\varepsilon(\varphi_1, \varphi_2)$ ,

$$\|u_{t_1}(\varphi_1) - u_{t_1}(\varphi_2)\|_1 = \|w_{t_1}(\varphi_1) - w_{t_1}(\varphi_2)\|_1 \leq 4(t_1 - t_0)^{-\frac{1}{2}}\varepsilon,$$

by (4.5), we further have for all  $\varepsilon \in (0, \varepsilon_0)$ ,

$$\begin{aligned}
\|\mathcal{P}_{t_2}(\varphi_1, \cdot) - \mathcal{P}_{t_2}(\varphi_2, \cdot)\|_{\text{TV}} &\leq \left( 4K_1(t_2 - t_1)^{-\frac{1}{\alpha} - \frac{\theta-1}{2}}(t_1 - t_0)^{-\frac{1}{2}}\varepsilon + \frac{K_2(t_2 - t_1)^{\frac{1}{\alpha}}}{R} \right) \\
&\quad \times \mathbb{P}(A_{t_1}^\varepsilon(\varphi_1, \varphi_2)) + 2(1 - \mathbb{P}(A_{t_1}^\varepsilon(\varphi_1, \varphi_2))) \\
&= 2 - \left( 2 - 4K_1(t_2 - t_1)^{-\frac{1}{\alpha} - \frac{\theta-1}{2}}(t_1 - t_0)^{-\frac{1}{2}}\varepsilon - \frac{K_2(t_2 - t_1)^{\frac{1}{\alpha}}}{R} \right) \\
&\quad \times \mathbb{P}(A_{t_1}^\varepsilon(\varphi_1, \varphi_2)).
\end{aligned}$$

Choosing first  $t_2 \in (t_1, t_0 + 1)$  so that

$$\frac{K_2(t_2 - t_1)^{\frac{1}{\alpha}}}{R} \leq \frac{1}{2},$$

and then  $\varepsilon \in (0, \varepsilon_0)$  so that

$$4K_1(t_2 - t_1)^{-\frac{1}{\alpha} - \frac{\theta-1}{2}}(t_1 - t_0)^{-\frac{1}{2}}\varepsilon \leq \frac{1}{2},$$

we finally obtain that for all  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}}$ ,

$$\|\mathcal{P}_{t_2}(\varphi_1, \cdot) - \mathcal{P}_{t_2}(\varphi_2, \cdot)\|_{\text{TV}} \leq 2 - \mathbb{P}(A_{t_1}^\varepsilon(\varphi_1, \varphi_2)) \stackrel{(4.13)}{\leq} 2 - \mathbb{P}(\Omega_{t_0}^\varepsilon).$$

The condition (ii) of Theorem 4.1 is thus verified by (2.10), and (iii) follows by Theorem 4.1. The whole proof is complete.  $\square$

## 5. APPENDIX: A STUDY OF DETERMINISTIC BURGERS EQUATION

In this appendix we study the following deterministic Burgers equation:

$$\dot{w}_t = -Aw_t - B(w_t + Z_t), \quad w_0 = \varphi \in \mathbb{H}^0, \quad (5.1)$$

where  $t \mapsto Z_t$  is a bounded measurable function on  $\mathbb{H}^1$ .

Recall the following estimate about the bilinear form  $B(u, v)$  (see [27, Lemma 2.1]):

$$\langle B(u, v), w \rangle_0 \leq C\|u\|_{\sigma_1}\|v\|_{\sigma_2+1}\|w\|_{\sigma_3}, \quad \sigma_1 + \sigma_2 + \sigma_3 > 1/2, \quad (5.2)$$

where  $C$  only depends on  $\sigma_1, \sigma_2, \sigma_3$ . Let  $\mathbb{M}_T$  be the Banach space defined by

$$\mathbb{M}_T := \left\{ u \in C([0, T]; \mathbb{H}) \cap C((0, T]; \mathbb{H}^1) : \|u\|_{\mathbb{M}_T} := \sup_{t \in [0, T]} (\|u_t\|_0 \vee (t^{\frac{1}{2}}\|u_t\|_1)) < +\infty \right\}.$$

We have

**Theorem 5.1.** For given  $R > 0$ , there exists a time  $T = T(R) \in (0, 1]$ , which is increasing as  $R \downarrow 0$ , such that if  $\sup_{t \in [0, T]} \|Z_t\|_1 \leq R$ , then

(i) for any  $\varphi \in \mathbb{B}_R^{\mathbb{H}}$ , there is a unique  $w = w(\varphi) \in \mathbb{B}_{2R}^{\mathbb{M}_T}$  satisfying that for all  $t \in [0, T]$ ,

$$w_t = e^{-tA} \varphi - \int_0^t e^{-(t-s)A} B(w_s + Z_s) ds; \quad (5.3)$$

(ii) for any  $\varphi_1, \varphi_2 \in \mathbb{B}_R^{\mathbb{H}}$ ,

$$\|w_*(\varphi_1) - w_*(\varphi_2)\|_{\mathbb{M}_T} \leq 2\|\varphi_1 - \varphi_2\|_0; \quad (5.4)$$

(iii) for any  $\varphi \in \mathbb{B}_R^{\mathbb{H}^1}$  and  $t \in [0, T]$ ,

$$\|w_t(\varphi)\|_1 \leq 3R. \quad (5.5)$$

Moreover, there are two constants  $C_1, C_2 > 0$  such that for any  $\varphi \in \mathbb{H}$  and all  $t \geq 0$ ,

$$\|w_t\|_0^2 \leq \|\varphi\|_0^2 e^{\int_0^t (C_1 \|Z_s\|_1^2 - 1) ds} + C_2 \int_0^t e^{\int_s^t (C_1 \|Z_r\|_1^2 - 1) dr} \|Z_s\|_1^4 ds. \quad (5.6)$$

In particular, for any  $\varphi \in \mathbb{H}$ , there exists a unique  $w_*(\varphi) \in \cup_{T>0} \mathbb{M}_T$  satisfying (5.3).

*Proof.* We use the fixed point argument. Fix  $\varphi \in \mathbb{B}_R^{\mathbb{H}}$ . Define a nonlinear map on  $\mathbb{M}_T$  by

$$\mathcal{M}(w)_t := e^{-tA} \varphi - \int_0^t e^{-(t-s)A} B(w_s + Z_s) ds.$$

We want to show that for some  $T := T(R) \leq 1$ ,

$\mathcal{M}$  is a contraction operator on  $\mathbb{B}_{2R}^{\mathbb{M}_T}$ .

Fix  $\sigma \in (\frac{1}{2}, 1)$ . For  $w \in \mathbb{B}_{2R}^{\mathbb{M}_T}$ , by (2.1) and (5.2), we have for all  $t \leq T$ ,

$$\begin{aligned} \|\mathcal{M}(w)_t\|_0 &\leq \|\varphi\|_0 + C_\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} \|B(w_s + Z_s)\|_{-\sigma} ds \\ &\leq R + C_\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} (\|w_s\|_0 + \|Z_s\|_0) (\|w_s\|_1 + \|Z_s\|_1) ds \\ &\leq R + C_\sigma R \int_0^t (t-s)^{-\frac{\sigma}{2}} (s^{-\frac{1}{2}} R + R) ds \\ &\leq R + C_\sigma R \left( t^{\frac{1-\sigma}{2}} R + t^{1-\frac{\sigma}{2}} R \right) \leq R + C_\sigma R^2 t^{\frac{1-\sigma}{2}}, \end{aligned}$$

where  $C_\sigma$  only depends on  $\sigma$ . Similarly, we also have

$$\begin{aligned} \|\mathcal{M}(w)_t\|_1 &\leq t^{-\frac{1}{2}} \|\varphi\|_0 + C_\sigma \int_0^t (t-s)^{-\frac{1+\sigma}{2}} \|B(w_s + Z_s)\|_{-\sigma} ds \\ &\leq t^{-\frac{1}{2}} R + C_\sigma R^2 \int_0^t (t-s)^{-\frac{1+\sigma}{2}} s^{-\frac{1}{2}} ds \\ &\leq t^{-\frac{1}{2}} R + C_\sigma R^2 t^{-\frac{\sigma}{2}}. \end{aligned}$$

Hence,

$$\|\mathcal{M}(w)\|_{\mathbb{M}_T} \leq R + C_\sigma R^2 T^{\frac{1-\sigma}{2}}.$$

If we choose  $T \leq (C_\sigma R)^{-\frac{2}{1-\sigma}} \wedge 1 =: T_1$ , then  $\|\mathcal{M}(w)\|_{\mathbb{M}_T} \leq 2R$ , and

$$\mathcal{M} \text{ maps } \mathbb{B}_{2R}^{\mathbb{M}_T} \text{ into } \mathbb{B}_{2R}^{\mathbb{M}_T}.$$

On the other hand, for  $w, v \in \mathbb{B}_{2R}^{\mathbb{M}_T}$ , by (5.2) again, we have

$$\begin{aligned}
\|\mathcal{M}(w)_t - \mathcal{M}(v)_t\|_0 &\leq C_\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} \|B(w_s + Z_s) - B(v_s + Z_s)\|_{-\sigma} ds \\
&\leq C_\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} \|w_s - v_s\|_0 (\|w_s\|_1 + \|Z_s\|_1) ds \\
&\quad + C_\sigma \int_0^t (t-s)^{-\frac{\sigma}{2}} \|w_s - v_s\|_1 (\|v_s\|_0 + \|Z_s\|_0) ds \\
&\leq C_\sigma \sup_{s \in [0, t]} \|w_s - v_s\|_0 \int_0^t (t-s)^{-\frac{\sigma}{2}} (s^{-\frac{1}{2}} R + R) ds \\
&\quad + C_\sigma \sup_{s \in [0, t]} s^{\frac{1}{2}} \|w_s - v_s\|_1 \int_0^t (t-s)^{-\frac{\sigma}{2}} s^{-\frac{1}{2}} R ds \\
&\leq C_\sigma R \|w - v\|_{\mathbb{M}_t} t^{\frac{1-\sigma}{2}},
\end{aligned}$$

and

$$\|\mathcal{M}(w)_t - \mathcal{M}(v)_t\|_1 \leq C_\sigma R \|w - v\|_{\mathbb{M}_t} t^{-\frac{\sigma}{2}}.$$

Hence,

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbb{M}_T} \leq C_\sigma R \|w - v\|_{\mathbb{M}_T} T^{\frac{1-\sigma}{2}}.$$

Letting  $T \leq \frac{1}{2C_\sigma R} \wedge T_1 =: T_2$ , we obtain

$$\|\mathcal{M}(w) - \mathcal{M}(v)\|_{\mathbb{M}_T} \leq \frac{1}{2} \|w - v\|_{\mathbb{M}_T}. \quad (5.7)$$

The existence and uniqueness for equation (5.3) follow by the fixed point theorem. Moreover, as in estimating (5.7), we also have (5.4).

Next we prove (5.5). As above, by (2.1) and (5.2), we have

$$\begin{aligned}
\|w_t\|_1 &\leq \|e^{-tA} \varphi\|_1 + \int_0^t \|e^{-(t-s)A} B(w_s + Z_s)\|_1 ds \\
&\leq \|\varphi\|_1 + C_\sigma \int_0^t (t-s)^{-\frac{1+\sigma}{2}} \|B(w_s + Z_s)\|_{-\sigma} ds \\
&\leq \|\varphi\|_1 + C_\sigma \int_0^t (t-s)^{-\frac{1+\sigma}{2}} (\|w_s\|_0 + \|Z_s\|_0) (\|w_s\|_1 + \|Z_s\|_1) ds \\
&\leq \|\varphi\|_1 + C_\sigma R \int_0^t (t-s)^{-\frac{1+\sigma}{2}} (\|w_s\|_1 + R) ds \\
&\leq \|\varphi\|_1 + C_\sigma R t^{\frac{1-\sigma}{2}} \left( \sup_{s \in [0, t]} \|w_s\|_1 + R \right).
\end{aligned}$$

From this, one sees that for  $t \leq (2(C_\sigma R))^{-\frac{2}{1-\sigma}} \wedge T_2$ ,

$$\sup_{s \in [0, t]} \|w_s\|_1 \leq 2\|\varphi\|_1 + 2C_\sigma R^2 t^{\frac{1-\sigma}{2}} \leq 3R.$$

We now prove (5.6). Notice that

$$\partial_t \|w_t\|_0^2 = -2\|w_t\|_1^2 + 2\langle B(w_t + Z_t), w_t \rangle_0.$$

Since  $\langle B(w, w), w \rangle_0 = 0$ , we have

$$\langle B(w_t + Z_t), w_t \rangle_0 = \langle B(w_t, Z_t), w_t \rangle_0 + \langle B(Z_t, Z_t), w_t \rangle_0 + \langle B(Z_t, w_t), w_t \rangle_0.$$

Thus, by (5.2) and Young's inequality, we have

$$2|\langle B(w_t + Z_t), w_t \rangle_0| \leq C\|w_t\|_1 \|Z_t\|_1 \|w_t\|_0 + C\|Z_t\|_1^2 \|w_t\|_1$$

$$\leq \|w_t\|_1^2 + C_1 \|w_t\|_0^2 \|Z_t\|_1^2 + C_2 \|Z_t\|_1^4.$$

Hence, by  $\|w\|_0 \leq \|w\|_1$ , we obtain

$$\partial_t \|w_t\|_0^2 \leq (C_1 \|Z_t\|_1^2 - 1) \|w_t\|_0^2 + C_2 \|Z_t\|_1^4,$$

which implies (5.6) by solving this differential inequality.  $\square$

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